Spectrum of a non-self-adjoint operator associated with the periodic heat equation

Marina Chugunova and Dmitry Pelinovsky
Department of Mathematics, McMaster University, Hamilton, Ontario, Canada, L8S 4K1

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Abstract

We study the spectrum of the linear operator $L = -\partial_\theta - \epsilon\partial_\theta(\sin\theta\partial_\theta)$ subject to the periodic boundary conditions on $\theta \in [-\pi, \pi]$. We prove that the operator is closed in $L^2([-\pi, \pi])$ with the domain in $H^1_{\rm per}([-\pi, \pi])$ for $|\epsilon| < 2$, its spectrum consists of an infinite sequence of isolated eigenvalues and the set of corresponding eigenfunctions is complete. By using numerical approximations of eigenvalues and eigenfunctions, we show that all eigenvalues are simple, located on the imaginary axis and the angle between two subsequent eigenfunctions tends to zero for larger eigenvalues. As a result, the complete set of linearly independent eigenfunctions does not form a basis in $H^1_{\rm per}([-\pi, \pi])$.

1 Introduction

We address the Cauchy problem for the periodic heat equation

$$\left\{ \begin{array}{l} \dot{h} = -h_\theta - \epsilon(\sin\theta h_\theta)_\theta, \quad t > 0, \\ h(0) = h_0, \end{array} \right. \tag{1.1}$$

subject to the periodic boundary conditions on $\theta \in [-\pi, \pi]$. This model was derived in the context of the dynamics of a thin viscous fluid film on the inside surface of a cylinder rotating around its axis in [3]. Extension of the model to the three-dimensional motion of the film was reported in [4].

The parameter ϵ is small for applications in fluid dynamics [3] and our main results cover the interval $|\epsilon| < 2$ in accordance to these applications. For any $\epsilon > 0$, the Cauchy problem for the heat equation (1.1) on the half-interval $\theta \in [0, \pi]$ is generally ill-posed [13] and it is naturally to expect that the Cauchy problem remains ill-posed on the entire interval $\theta \in [-\pi, \pi]$. The authors of the pioneer work [3] used a heuristic asymptotic solution to suggest that the growth of "explosive instabilities" might occur in the time evolution of the Cauchy problem (1.1).

Nevertheless, in a contradiction with the picture of explosive instabilities, only purely imaginary eigenvalues were discovered in the discrete spectrum of the associated linear operator

$$L = -\epsilon \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) - \frac{\partial}{\partial\theta}, \tag{1.2}$$

acting on sufficiently smooth periodic functions $f(\theta)$ on $\theta \in [-\pi, \pi]$. Various approximations of eigenvalues were obtained in [3] by two asymptotic methods (expansions in powers of ϵ and the WKB method)

and by three numerical methods (the Fourier series approximations, the pseudospectral method, and the Newton–Raphson iterations). The results of the pseudospectral method were checked independently in [16] (see pp. 124–125 and 406–408). It is seen both in [3] and [16] that the level sets of the resolvent of $(\lambda - L)^{-1}$ form divergent curves to the left and right half-planes and, while true eigenvalues lie on the imaginary axis, eigenvalues of the truncated Fourier series may occur in the left and right half-planes of the spectral plane. This distinctive feature was interpreted in [3] towards the picture of growth of disturbances and the phenomenon of explosive instability.

One more question raised in [3] was about the validity of the series of eigenfunctions associated to the purely imaginary eigenvalues of the operator L for $\epsilon \neq 0$. Although various initial conditions h_0 were decomposed into a finite sum of eigenfunctions and the error decreased with a larger number of terms in the finite sum, the authors of [3] conjectured that the convergence of the series depended on the time variable and "even though the series converges at $t = 0$, it may diverge later". This conjecture would imply that the eigenfunctions of L for $\epsilon \neq 0$ do not form a basis of functions in the space $H^s([-\pi, \pi])$ with $s > \frac{1}{2}$ unlike the harmonics of the complex Fourier series associated with the operator L for $\epsilon = 0$.

In this paper, we prove that the operator L is closed in $L^2([-\pi, \pi])$ with a domain in $H^1_{\rm per}([-\pi, \pi])$ for $|\epsilon| < 2$, such that the spectrum of the eigenvalue problem

$$-\epsilon \frac{d}{d\theta}\left(\sin\theta \frac{df}{d\theta}\right) - \frac{df}{d\theta} = \lambda f, \qquad f \in H^1_{\rm per}([-\pi, \pi]), \tag{1.3}$$

is well-defined. Here and henceforth, we denote

$$H^1_{\rm per}([-\pi, \pi]) = \left\{ f \in H^1([-\pi, \pi]) : \quad f(\pi) = f(-\pi) \right\}. \tag{1.4}$$

Furthermore, we prove that the residual and essential spectra of the spectral problem (1.3) are empty and the eigenvalues of the discrete spectrum accumulate at infinity along the imaginary axis. We further prove completeness of the series of eigenfunctions associated to all eigenvalues of the discrete spectrum of L in $H^1_{\rm per}([-\pi, \pi])$. Using the numerical approximations of eigenvalues and eigenfunctions of the spectral problem (1.3), we show that all eigenvalues of L are simple, located at the imaginary axis, and the angle between two subsequent eigenfunctions tends to zero for larger eigenvalues. As a result, the complete set of linearly independent eigenfunctions does not form a basis in $H^1_{\rm per}([-\pi, \pi])$ and hence it cannot be used directly in the studies of well-posedness of the Cauchy problem associated with the heat equation (1.1).

The paper is structured as follows. The domain, closed extensions, and the essential spectrum of the operator L are analyzed in Section 2. The discrete spectrum of the operator L is characterized in Section 3. Section 4 presents numerical approximations of eigenvalues and eigenfunctions of the spectral problem (1.3). Section 5 gives a summary and discusses the Cauchy problem for the heat equation (1.1). Appendix A reports an extension of the spectral problem (1.3) into a self-adjoint problem in a weighted L^2-space. Appendix B reformulates the eigenvalue problem (1.3) as the resonance pole problem for a linear Schrödinger operator on an infinite line. These Appendices are not related to our main results and are left in the text for future work and references.

2 General properties of the linear operator L

It is obvious that the operator L is densely defined in $L^2([-\pi, \pi])$ on the space of smooth functions with periodic boundary conditions. However, the operator L is not closed in $L^2([-\pi, \pi])$ if the functions

are infinitely smooth. We shall therefore prove that the operator L admits a closure in $L^2([-\pi,\pi])$ with a domain in $H^1_{\rm per}([-\pi,\pi])$ (Lemmas 1 and 2). Properties of the eigenfunctions and eigenvalues of the spectral problem (1.3) in $H^1_{\rm per}([-\pi,\pi])$ are then studied (Lemmas 3 and 4) and the absence of the residual and essential spectra is rigorously proved (Lemmas 5 and 6).

Lemma 1 *If $Lf = F$ and $F \in L^2([-\pi,\pi])$ then $f \in H^1_{\rm per}([-\pi,\pi])$ for $|\epsilon| < 2$.*

Proof. For simplicity, we will set $\epsilon > 0$. The case $\epsilon < 0$ is considered similarly. Let $f(\theta)$ be in the domain of L on $\theta \in [-\pi,\pi]$, such that $Lf = F$ for any $F \in L^2([-\pi,\pi])$. By the method of variation of constants, we express $f'(\theta)$ from the solution of the first-order ODE for $f'(\theta)$:

$$f'(\theta) = \frac{\cot^{1/\epsilon}(\theta/2)}{\sin\theta}\left[\int_{\theta_0}^{\theta} \tan^{1/\epsilon}(\theta_1/2) F(\theta_1) d\theta_1 + C\right], \tag{2.1}$$

where C is an integration constant for any $\theta_0 \in [-\pi,\pi]$. The function $f'(\theta)$ is continuously differentiable at any $\theta \in (-\pi,0)\cup(0,\pi)$. Let us consider the behavior of $f'(\theta)$ at the regular singular points $\theta_0 = 0$ and $\theta_0 = \pi$ on $\theta \in [0,\pi]$. A similar consideration holds at the regular singular points $\theta_0 = 0$ and $\theta_0 = -\pi$ on $\theta \in [-\pi,0]$. For $\theta_0 = 0$ and sufficiently small $\theta > 0$, it follows from (2.1) that

$$\left|\int_0^{\theta} \tan^{1/\epsilon}(\theta_1/2) F(\theta_1) d\theta_1\right|^2 \le \int_0^{\theta} \tan^{2/\epsilon}(\theta_1/2) d\theta_1 \int_0^{\theta} F^2(\theta_1) d\theta_1 \le \alpha_1^2(\theta)\theta^{2/\epsilon+1},$$

where $\lim_{\theta\to 0^+} \alpha_1(\theta) = 0$. As a result, for $\theta > 0$, it holds that

$$|f'(\theta)| \le \frac{\alpha_1(\theta)}{\theta^{1/2}} + \frac{C\alpha_2(\theta)}{\theta^{1/\epsilon+1}}, \tag{2.2}$$

where $\lim_{\theta\to 0^+} \alpha_2(\theta) \neq 0$. If $\epsilon < 2$, $f' \in L^2$ near $\theta = 0$ if and only if $C = 0$.

For $\theta_0 = \pi$ and sufficiently small $\pi - \theta > 0$, it follows from (2.1) that

$$\begin{aligned}\left|\int_{\pi}^{\theta} \tan^{1/\epsilon}(\theta_1/2) F(\theta_1) d\theta_1\right|^2 &\le \int_{\theta}^{\pi} (\pi-\theta_1)^{-2/\epsilon} d\theta_1 \int_{\theta}^{\pi} (\pi-\theta)^{2/\epsilon} \tan^{2/\epsilon}(\theta_1/2) F^2(\theta_1) d\theta_1 \\ &\le \alpha_1^2(\theta)(\pi-\theta)^{-2/\epsilon+1},\end{aligned}$$

where $\lim_{\theta\to\pi^-} \alpha_1(\theta) = 0$. As a result, for $\theta < \pi$, it holds that

$$|f'(\theta)| \le \frac{\alpha_1(\theta)}{(\pi-\theta)^{1/2}} + C\alpha_2(\theta)(\pi-\theta)^{1/\epsilon-1}, \tag{2.3}$$

where $\lim_{\theta\to\pi^-} \alpha_2(\theta) \neq 0$. If $\epsilon < 2$, $f' \in L^2$ near $\theta = \pi$ for any $C \neq 0$. Finally, we recall the Neumann–Poincare inequality on $\theta \in [-\pi,\pi]$:

$$\|f\|_{L^2}^2 \le 4\pi^2 \|f'\|_{L^2}^2 + \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(\theta) d\theta\right)^2. \tag{2.4}$$

Due to the estimates (2.2) and (2.3), we can see that $\|f\|_{L^1}$ is bounded on $\theta \in [-\pi,\pi]$ and so is $\|f\|_{L^2}$. Therefore, the solution $f(\theta)$ of $Lf = F$ with $F \in L^2([-\pi,\pi])$ lies in $H^1([-\pi,\pi])$ if $|\epsilon| < 2$. $\square$

Lemma 2 *The operator L admits a closure in $L^2([-\pi,\pi])$ for $|\epsilon| < 2$ with $Dom(L) \subset H^1_{\rm per}([-\pi,\pi])$.*

Proof. According to Lemma 1.1.2 in [6], if an operator has a non-empty spectrum in a proper subset of a complex plane, then it must be closed. The operator L has a non-empty spectrum in $H^1_{\rm per}([-\pi,\pi])$ since $\lambda = 0$ is an eigenvalue with the eigenfunction $f_0(\theta) = 1 \in H^1_{\rm per}([-\pi,\pi])$. We should show that there exists at least one regular point $\lambda_0 \in \mathbb{C}$, such that

$$\forall f \in H^1_{\rm per}([-\pi,\pi]) : \quad \|(L-\lambda_0 I)f\|_{L^2} \geq k_0 \|f\|_{L^2} \tag{2.5}$$

for some $k_0 > 0$. In particular, we show that any $\lambda_0 \in \mathbb{R}$ is a regular point of L in $H_0 \subset H^1_{\rm per}([-\pi,\pi])$, where

$$H_0 = \left\{ f \in H^1_{\rm per}([-\pi,\pi]) : \quad \int_{-\pi}^{\pi} f(\theta) d\theta = 0 \right\}. \tag{2.6}$$

By using straightforward computations, we obtain

$$(f', Lf) = -\int_{-\pi}^{\pi} (1+\epsilon\cos\theta) |f'|^2 d\theta - \epsilon \int_{-\pi}^{\pi} \sin\theta \bar{f}' f'' d\theta, \tag{2.7}$$

where $(g,f) = \int_{-\pi}^{\pi} \bar{g}(\theta) f(\theta) d\theta$ is a standard inner product in L^2. If $f \in H^1_{\rm per}([-\pi,\pi])$, then

$$\mathrm{Re}(f', f) = 0, \qquad \mathrm{Re}(f', Lf) = -\int_{-\pi}^{\pi} \left(1 + \frac{\epsilon}{2}\cos\theta\right) |f'|^2 d\theta, \tag{2.8}$$

such that for any $\lambda_0 \in \mathbb{R}$ it is true that

$$|\mathrm{Re}(f', (L-\lambda_0 I)f)| \geq \left(1 - \frac{|\epsilon|}{2}\right) \|f'\|^2_{L^2}.$$

By using the Cauchy–Schwarz inequality, we estimate the left-hand-side term from above

$$|\mathrm{Re}(f', (L-\lambda_0 I)f)| \leq |(f', (L-\lambda_0 I)f)| \leq \|f'\|_{L^2} \|(L-\lambda_0 I)f\|_{L^2},$$

such that

$$\|(L-\lambda_0 I)f\|_{L^2} \geq \left(1 - \frac{|\epsilon|}{2}\right) \|f'\|_{L^2}. \tag{2.9}$$

Using the Neumann–Poincare inequality (2.4) for any $f \in H_0 \subset H^1_{\rm per}([-\pi,\pi])$, we continue the right-hand-side of the inequality (2.9) and recover the inequality (2.5) for any $\lambda_0 \in \mathbb{R}$ with

$$k_0 = \frac{1}{2\pi}\left(1 - \frac{|\epsilon|}{2}\right) > 0.$$

The estimate holds if $|\epsilon| < 2$. $\square$

Remark 1 The formal adjoint of L in $L^2([-\pi,\pi])$ is $L^* = -\epsilon\partial_\theta(\sin\theta\partial_\theta) + \partial_\theta$. According to Lemma 1.2.1 in [6], the operator L^* also admits a closure in $L^2([-\pi,\pi])$ with $\mathrm{Dom}(L^*) \subset H^1_{\rm per}([-\pi,\pi])$ for $|\epsilon| < 2$.

Lemma 3 *Let λ be a isolated eigenvalue of the spectral problem $Lf = \lambda f$ with an eigenfunction $f \in H^1_{\rm per}([-\pi,\pi])$. Then,*

(i) $-\lambda$, $\bar{\lambda}$ *and* $-\bar{\lambda}$ *are also eigenvalues of the spectral problem* $Lf = \lambda f$ *with the eigenfunctions* $f(-\theta)$, $\bar{f}(\theta)$ *and* $\bar{f}(-\theta)$ *in* $H^1_{\rm per}([-\pi,\pi])$.

(ii) λ *is also an eigenvalue of the adjoint spectral problem* $L^* f^* = \lambda f^*$ *with the eigenfunction* $f^* = f(\pi - \theta)$ *in* $H^1_{\rm per}([-\pi,\pi])$.

(iii) λ *is a simple eigenvalue of* $Lf = \lambda f$ *if and only if* $(f^*, f) \neq 0$.

Proof. (i) Due to inversion $\theta \to -\theta$, the spectral problem (1.3) transforms to itself with the transformation $\lambda \to -\lambda$. Due to the complex conjugation, it transforms to itself with $\lambda \to \bar{\lambda}$. (ii) Due to the transformation $\theta \to \pi - \theta$, the spectral problem (1.3) transforms to the adjoint problem $L^* f = \lambda f$ with the same eigenvalue. (iii) The assertion follows by the Fredholm Alternative Theorem for isolated eigenvalues. $\square$

Lemma 4 *Let* λ *be an eigenvalue of the spectral problem (1.3) with the eigenfunction* $f \in H^1_{\rm per}([-\pi,\pi])$. *Then,*

$$\mathrm{Re}(\lambda) = \epsilon \frac{(f', \sin\theta f')}{(f,f)}, \qquad i\mathrm{Im}(\lambda) = \frac{(f',f)}{(f,f)}, \tag{2.10}$$

and $\mathrm{Im}(\lambda) \neq 0$ *except for a simple zero eigenvalue* $\lambda = 0$.

Proof. By constructing the quadratic form for $f \in H^1_{\rm per}([-\pi,\pi])$, we obtain

$$(f, Lf) = \epsilon \int_{-\pi}^{\pi} \sin\theta |f'|^2 d\theta - \int_{-\pi}^{\pi} \bar{f} f' d\theta, \tag{2.11}$$

where the second term is purely imaginary since

$$f \in H^1_{\rm per}([-\pi,\pi]) : \quad \int_{-\pi}^{\pi} \bar{f}' f d\theta = |f(\theta)|^2 |_{\theta=-\pi}^{\theta=\pi} - \int_{-\pi}^{\pi} \bar{f} f' d\theta = -\overline{\int_{-\pi}^{\pi} \bar{f}' f d\theta}. \tag{2.12}$$

Moreover, the equality (2.8) can be rewritten in the form

$$i\mathrm{Im}(\lambda)(f',f) = \mathrm{Re}(f', Lf) = -\int_{-\pi}^{\pi} \left(1 + \frac{\epsilon}{2}\cos\theta\right) |f'(\theta)|^2 d\theta \leq -\left(1 - \frac{|\epsilon|}{2}\right) \|f'\|^2_{L^2}, \tag{2.13}$$

where the right-hand side is negative if $|\epsilon| < 2$ and $f(\theta)$ is not constant on $\theta \in [-\pi,\pi]$. Therefore, $(f',f) \neq 0$ and $\mathrm{Im}(\lambda) \neq 0$. Finally, the constant eigenfunction $f(\theta) = 1$ corresponds to the eigenvalue $\lambda = 0$ and it is a simple eigenvalue since $(f^*, f) \neq 0$, where $f^*(\theta) = f(\pi - \theta) = 1$ is an eigenfunction of the adjoint operator L^* for the same eigenvalue $\lambda = 0$. $\square$

Lemma 5 *The residual spectrum of the operator* L *in* $L^2([-\pi,\pi])$ *is empty.*

Proof. By a contradiction, assume that λ belongs to the residual part of the spectrum of L such that $\mathrm{Ker}(L - \lambda I) = \varnothing$ but $\mathrm{Range}(L - \lambda I)$ is not dense in $L^2([-\pi,\pi])$. Let $g \in L^2([-\pi,\pi])$ be orthogonal to $\mathrm{Range}(L - \lambda I)$, such that

$$\forall f \in L^2([-\pi,\pi]) : \quad 0 = (g, (L - \lambda I) f) = ((L^* - \bar{\lambda} I) g, f).$$

Therefore, $(L^* - \bar{\lambda} I) g = 0$, that is $\bar{\lambda}$ is an eigenvalue of L^*. By Lemma 3(ii), $\bar{\lambda}$ is an eigenvalue of L and by Lemma 3(i), λ is also an eigenvalue of L. Hence λ can not be in the residual part of the spectrum of L. $\square$

Lemma 6 *The essential spectrum of the operator L in $L^2([-\pi,\pi])$ is empty.*

Proof. According to Theorem 4 on p.1438 in [8], if L is a differential operator defined on the interval $\theta \in (-\pi,\pi) = (-\pi,0) \cup (0,\pi)$ and $L_\pm$ are restrictions of L on $\theta \in (-\pi,0)$ and $\theta \in (0,\pi)$, then $\sigma_e(L) = \sigma_e(L_+) \cup \sigma_e(L_-)$, where $\sigma_e(L)$ denotes the essential spectrum of L. By the symmetry of the two intervals, it is sufficient to prove that the operator L_+ has no essential spectrum on $\theta \in (0,\pi)$ (independently of the boundary conditions at $\theta = 0$ and $\theta = \pi$). It is also sufficient to carry out the proof for $\epsilon > 0$.

Let us consider the spectral problem (1.3) on $\theta \in [0,\pi]$ and use the transformation

$$\cos\theta = \tanh t, \quad \sin\theta = \mathrm{sech} t, \qquad t \in \mathbb{R},$$

such that $\theta \in [0,\pi]$ is mapped to the infinite line $t \in \mathbb{R}$. Let $f_+(t) = f(\theta)$ on $\theta \in [0,\pi]$. The function $f_+(t)$ satisfies the spectral problem

$$-\epsilon f_+''(t) + f_+'(t) = \lambda \mathrm{sech} t \, f_+(t). \tag{2.14}$$

With a transformation $f_+(t) = e^{t/2\epsilon} g_+(t)$, the spectral problem (2.14) is written in the symmetric form

$$-\epsilon g_+''(t) + \frac{1}{4\epsilon} g_+(t) = \lambda \mathrm{sech} t \, g_+(t). \tag{2.15}$$

Thus, our operator is extended to a symmetric operator with an exponentially decaying weight $\rho(t) = \mathrm{sech}(t)$. According to Corollary 3 on p. 1437 in [8], if L is a symmetric operator on an open interval (a,b) and L_0 is a self-adjoint extension of L with respect to some boundary conditions at $x = a$ and $x = b$, then $\sigma_e(L) = \sigma_e(L_0)$. Here $a = -\infty$, $b = \infty$, and we need to show that the essential spectrum of the symmetric problem (2.15) is empty in $L^2(\mathbb{R})$. This follows by Theorem 7 on p.93 in [10]: since the weight function $\rho(t)$ of the problem $-y''(t) - \lambda\rho(t)y(t) = 0$ on $t \in \mathbb{R}$ decays faster than $1/t^2$ as $|t| \to \infty$, the spectrum of $-y''(t) - \lambda\rho(t)y(t) = 0$ is purely discrete[1]. □

3 The discrete spectrum of the linear operator L

By results of Lemmas 3, 4, 5, and 6, the spectral problem (1.3) for $|\epsilon| < 2$ may have only two types of spectral data besides the simple zero eigenvalue: either pairs of purely imaginary eigenvalues or quartets of symmetric complex eigenvalues. We shall prove that there exists an infinite sequence of eigenvalues λ which accumulate to infinity along the imaginary axis (Lemmas 7 and 8). We further prove completeness of the eigenfunctions associated to all isolated eigenvalues of the spectral problem (1.3) (Theorem 1). Finally, Theorem 2 formulates a sufficient condition for the set of eigenfunctions of the spectral problem (1.3) not to form a basis in $H^1_{\mathrm{per}}([-\pi,\pi])$.

Lemma 7 *Let $0 < \epsilon < 2$ and $\epsilon \neq \frac{1}{n}$, $n \in \mathbb{N}$. For $\lambda \in \mathbb{C}$, the spectral problem (1.3) admits three sets of linearly independent solutions $\{f_1(\theta), f_2(\theta)\}$ given by the Frobenius series*

$$-\pi < \theta < \pi: \quad f_1 = 1 + \sum_{n\in\mathbb{N}} c_n \theta^n, \quad f_2 = \theta^{-1/\epsilon}\left(1 + \sum_{n\in\mathbb{N}} d_n \theta^n\right), \tag{3.1}$$

[1]Although the spectral problem (2.15) has an additional term $Cy(t)$ with $C > 0$, this term only makes better the inequality (30) on p.93 in the proof of Theorem 7 of [10].

or

$$0<\pm\theta<\pi: \quad f_1^{\pm}=1+\sum_{n\in\mathbb{N}} a_n^{\pm}(\pi\mp\theta)^n, \quad f_2^{\pm}=(\pi\mp\theta)^{1/\epsilon}\left(1+\sum_{n\in\mathbb{N}} b_n^{\pm}(\pi\mp\theta)^n\right), \tag{3.2}$$

where all coefficients are uniquely defined. The solution $f_1(\theta)$ is an analytic function of $\lambda\in\mathbb{C}$ uniformly on $\theta\in[-\pi,\pi]$.

Proof. Existence of two linearly independent solutions on $-\pi<\theta<\pi$ in the form (3.1) and on $0<\pm\theta<\pi$ in the form (3.2) follows by the ODE analysis near the regular singular points [5]. The difference between the two indices of the indicial equation is $\frac{1}{\epsilon}$ and it is non-integer for $\epsilon\neq\frac{1}{n}$, $n\in\mathbb{N}$[2]. Since the spectral problem (1.3) depends analytically on λ and the Frobenius series converges absolutely and uniformly in between two regular singular points, the solution $f_1(\theta)$ is analytic in $\lambda\in\mathbb{C}$ for any fixed $\theta\in(-\pi,\pi)$. Due to uniqueness of the solutions of the ODE (1.3), the solution $f_1(\theta)$ can be equivalently represented by the other solutions

$$f_1(\theta)=A^{\pm}f_1^{\pm}(\theta)+B^{\pm}f_2^{\pm}(\theta), \quad 0<\pm\theta<\pi, \tag{3.3}$$

where $A^{\pm}$ and $B^{\pm}$ are some constants, while the functions $f_1^{\pm}(\theta)$ and $f_2^{\pm}(\theta)$ are analytic in $\lambda\in\mathbb{C}$ for any fixed $\pm\theta\in(0,\pi]$. By matching analytic solutions for any $\pm\theta\in(0,\pi)$, we find that $A^{\pm}$ and $B^{\pm}$ are analytic functions of $\lambda\in\mathbb{C}$, the Frobenius series for $f_1(\theta)$ converges absolutely and uniformly on $\theta\in[-\pi,\pi]$, and the solution $f_1(\theta)$ is an analytic function in $\lambda\in\mathbb{C}$ uniformly on $\theta\in[-\pi,\pi]$. ☐

Corollary 1 *There exists an analytic function $F_\epsilon(\lambda)$ on $\mathrm{Im}\lambda>0$, roots of which give isolated eigenvalues of the spectral problem (1.3) with the account of their multiplicity. The only accumulation point of isolated eigenvalues in the λ-plane may occur at infinity.*

Proof. The function $f\in H^1([-\pi,\pi])$ satisfies the spectral problem (1.3) if and only if $f(\theta)=C_0 f_1(\theta)$ on $\theta\in[-\pi,\pi]$, where $C_0=1$ thanks to the scaling invariance of homogeneous equations. By using the representation (3.3), we can find that $A^{\pm}=\lim_{\theta\to\pm\pi} f_1(\theta)$ are uniquely defined analytic functions in $\lambda\in\mathbb{C}$. The function $F_\epsilon(\lambda)=A^+-A^-$ is analytic function of $\lambda\in\mathbb{C}$ by construction and zeros of $F_\epsilon(\lambda)$ on $\mathrm{Im}\lambda>0$ coincide with the eigenvalues λ of the spectral problem (1.3) with the account of their multiplicity. If $F_\epsilon(\lambda_0)=0$ for some $\lambda_0\in\mathbb{C}$, the corresponding eigenfunction $f(\theta)$ lies in $H^1_{\mathrm{per}}([-\pi,\pi])$, i.e. it satisfies the periodic boundary conditions $f(\pi)=f(-\pi)$. By analytic function theory, the sequence of roots of $F_\epsilon(\lambda)$ can not accumulate at a finite point on $\lambda\in\mathbb{C}$. ☐

Remark 2 We will use the method involving the analytic function $F_\epsilon(\lambda)$ on $\lambda\in\mathbb{C}$ for a numerical shooting method which enables us to approximate eigenvalues of the spectral problem (1.3). This method involves less computations than the shooting method described in Appendix C of [3]. Nevertheless, it is essentially the same shooting method and it uses the ODE analysis near the regular singular point (Lemma 7), which repeats the arguments in Appendix B of [3].

Lemma 8 *Fix $0<\epsilon<2$ and let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a set of eigenvalues of the spectral problem (1.3) with $\mathrm{Im}\lambda_n>0$, ordered in the ascending order of $|\lambda_n|$. There exists a finite number $N\geq 1$, such that for all $n\geq N$, $\lambda_n=i\omega_n\in i\mathbb{R}_+$ and*

$$\omega_n=Cn^2+\mathrm{o}(n^2) \quad \text{as } n\to\infty, \tag{3.4}$$

for some $C>0$.

[2] An additional logarithmic term $\log(\pi-\theta)$ may need to be included into the Frobeneus series if $\epsilon=\frac{1}{n}$, $n\in\mathbb{Z}$.

Proof. We reduce the spectral problem (1.3) to two uncoupled Schroödinger equations on an infinite line. Let $f(\theta)$ be represented on two intervals $\pm\theta \in [0,\pi]$ by using the transformations

$$\cos\theta = \tanh t, \qquad \sin\theta = \pm \mathrm{sech} t, \tag{3.5}$$

where $t \in \mathbb{R}$. Then, the functions $f_\pm(t) = f(\theta)$ on $\pm\theta \in [0,\pi]$ satisfy the uncoupled spectral problems

$$-\epsilon f_\pm''(t) + f_\pm'(t) = \pm\lambda \mathrm{sech} t\, f_\pm(t), \quad t \in \mathbb{R}, \tag{3.6}$$

The normalization condition $f(0) = 1$ is equivalent to the condition $\lim_{t\to\infty} f_\pm(t) = 1$. The periodic boundary condition $f(\pi) = f(-\pi)$ is equivalent to the condition $\lim_{t\to-\infty} f_-(t) = \lim_{t\to-\infty} f_+(t)$. The linear problems (3.6) are reformulated as the quadratic Ricatti equations by using the new variables

$$f_\pm(t) = e^{\int_\infty^t S_\pm(t')dt'} : \quad S_\pm - \epsilon(S_\pm' + S_\pm^2) = \pm\lambda \mathrm{sech} t. \tag{3.7}$$

We choose a negative root of the quadratic equation in the form

$$S_\pm(t) = \frac{1 - \sqrt{1 \mp 4\epsilon\lambda \mathrm{sech} t - 4\epsilon^2 R_\pm}}{2\epsilon}, \qquad R_\pm = S_\pm'(t). \tag{3.8}$$

The representation (3.8) becomes the chain fraction if the derivative of $S_\pm(t)$ is defined recursively from the same expression (3.8). By using the theory of chain fractions, we claim that $R_\pm = \mathrm{O}(\sqrt{|\lambda|})$ as $|\lambda| \to \infty$ uniformly on $t \in \mathbb{R}$. The function $F_\epsilon(\lambda)$ of Corollary 1 is now expressed by

$$F_\epsilon(\lambda) = \lim_{t\to-\infty} [f_+(t) - f_-(t)] = e^{\int_{-\infty}^\infty S_+(t)dt} - e^{\int_{-\infty}^\infty S_-(t)dt}. \tag{3.9}$$

Zeros of $F_\epsilon(\lambda)$ are equivalent to zeros of the infinite set of functions

$$G_n(\lambda) = \frac{1}{4\pi i\epsilon}\int_{-\infty}^\infty \left[\sqrt{1 + 4\epsilon\lambda \mathrm{sech} t - 4\epsilon^2 R_-(t)} - \sqrt{1 - 4\epsilon\lambda \mathrm{sech} t - 4\epsilon^2 R_+(t)}\right] dt - n, \tag{3.10}$$

where $n \in \mathbb{N}$. If $R_\pm(t) \equiv 0$, the function $\tilde{G}_n(\omega) = G(i\omega)$, $n \in \mathbb{N}$ is real-valued and strictly increasing on $\omega \in \mathbb{R}_+$ with $\tilde{G}_n(0) = -n$. By performing asymptotic analysis, we compute that

$$\begin{aligned}
& \frac{1}{4\pi i\epsilon}\int_{-\infty}^\infty \left[\sqrt{1 + 4i\epsilon\omega \mathrm{sech} t - 4\epsilon^2 R_-(t)} - \sqrt{1 - 4i\epsilon\omega \mathrm{sech} t - 4\epsilon^2 R_+(t)}\right] dt \\
= \;& \frac{1}{\pi i}\int_{-\infty}^\infty \frac{2i\omega \mathrm{sech} t + \epsilon(R_+ - R_-)}{\sqrt{1 + 4i\epsilon\omega \mathrm{sech} t - 4\epsilon^2 R_-(t)} + \sqrt{1 - 4i\epsilon\omega \mathrm{sech} t - 4\epsilon^2 R_+(t)}} dt \\
= \;& \frac{\sqrt{\omega}}{\sqrt{2\epsilon}\pi}\int_{-\infty}^\infty \frac{dt}{\sqrt{\cosh t}} + \mathrm{o}\left(\sqrt{\omega}\right),
\end{aligned} \tag{3.11}$$

such that $\lim_{\omega\to\infty} \tilde{G}_n(\omega) = \infty$. Therefore, there exists exactly one root $\omega = \omega_n$ of $\tilde{G}_n(\omega)$ for each n. Since $R_- = \bar{R}_+$ for $\lambda = i\omega \in i\mathbb{R}$, each simple root of $\tilde{G}_n(\omega)$ persists for non-zero values of $R_\pm(t) = \mathrm{O}(\sqrt{\omega})$ uniformly on $t \in \mathbb{R}$ as $\omega \to \infty$. According to the asymptotic result (3.11), the roots ω_n of $\tilde{G}_n(\omega)$ satisfy the asymptotic distribution (3.4) with $C = \frac{2\epsilon\pi^2}{\left(\int_{-\infty}^\infty \frac{dt}{\sqrt{\cosh t}}\right)^2}$. $\square$

Remark 3 Analysis of Lemma 8 extends the formal WKB approach proposed in Section 3 of [3]. In particular, the equation (3.10) with $R_\pm = 0$ has been obtained in Eq. (3.11) of [3].

Theorem 1 *Let* $\{f_n(\theta)\}_{n\in\mathbb{N}}$ *be the set of eigenfunctions corresponding to the set of eigenvalues* $\{\lambda_n\}_{n\in\mathbb{N}}$ *in Lemma 8 with* $\mathrm{Im}\lambda_n > 0$*. The set of eigenfunctions is complete in a subspace of* $H_0 \subset H^1_{\mathrm{per}}([-\pi,\pi])$*.*

Proof. By Corollary 1, eigenvalues of L with $\mathrm{Im}\lambda > 0$ accumulate to infinity, such that the matrix operator $M = L^{-1}$ defined on $H_0 \subset H^1_{\mathrm{per}}([-\pi,\pi])$ is compact. By Lemma 8, there are infinitely many isolated eigenvalues and large eigenvalues are all purely imaginary, such that $|\lambda_n| = \mathrm{O}(n^2)$ as $n \to \infty$. These two facts satisfy two sufficient conditions of the Lidskii's Completeness Theorem. According to Theorem 6.1 on p. 302 in [11], the set of eigenvectors and generalized eigenvectors of a compact operator M in a Hilbert space H is complete if there exists $p > 0$ such that

$$s_n(M) = o(n^{\frac{-1}{p}}), \quad \text{as } n \to \infty, \tag{3.12}$$

where s_n is a singular number of the operator M, and the set

$$W_M = \{(Mf, f) : \quad f \in H, \quad \|f\|_{L^2} = 1\} \tag{3.13}$$

lies in a closed angle θ_M with vertex at 0 and opening $\frac{\pi}{p}$.

Since the singular numbers s_n are eigenvalues of the positive self-adjoint operator $(MM^*)^{1/2}$ and the eigenvalues of L grow like $\mathrm{O}(n^2)$ as $n \to \infty$, we have $s_n(M) = \mathrm{O}(n^{-2})$ as $n \to \infty$, such that the first condition (3.12) is verified with $p = 1$. Since all $\mathrm{Im}\lambda_n > 0$ for the set of eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ of Lemma 8, the spectrum of M lies in the lower half plane, such that the second condition (3.13) is also verified with $p = 1$ $(\theta_M = \pi)$. $\square$

Corollary 2 *The set of eigenfunctions* $\{f_n(\theta)\}_{n\in\mathbb{Z}}$ *with* $f_0 = 1$ *and* $f_{-n} = \bar{f}_n$*,* $\forall n \in \mathbb{N}$ *is complete in* $H^1_{\mathrm{per}}([-\pi,\pi])$*.*

Remark 4 Due to linear independence of eigenfunctions for distinct eigenvalues, the set of eigenfunctions $\{f_n(\theta)\}_{n\in\mathbb{Z}}$ is also minimal if all eigenvalues are simple[3]. If the set $\{f_n(\theta)\}_{n\in\mathbb{Z}}$ is complete and minimal, any function $f \in H^1_{\mathrm{per}}([-\pi,\pi])$ can be approximated by a finite linear combination $f_N(\theta) = \sum\limits_{n=-N}^{N} c_n f_n(\theta)$, such that for any $\varepsilon > 0$, there exists $N \geq 1$ and the set of coefficients $\{c_n\}_{-N\leq n\leq N}$, such that the inequality $\|f - f_N\|_{H^1_{\mathrm{per}}([-\pi,\pi])} < \epsilon$ holds. This approximation does not imply that the set $\{f_n(\theta)\}_{n\in\mathbb{Z}}$ forms a Schauder basis in $H^1_{\mathrm{per}}([-\pi,\pi])$, in which case there would exist a unique series representation $f(\theta) = \sum\limits_{n\in\mathbb{Z}} c_n f_n(\theta)$ for any $f \in H^1_{\mathrm{per}}([-\pi,\pi])$, such that $\lim\limits_{N\to\infty} \|f - f_N\|_{H^1_{\mathrm{per}}([-\pi,\pi])} = 0$.

Theorem 2 *Let* $\{f_n(\theta)\}_{n\in\mathbb{Z}}$ *be a complete and minimal set of eigenfunctions of the spectral problem (1.3) for the set of eigenvalues* $\{\lambda_n\}_{n\in\mathbb{Z}}$ *in Theorem 1. The set of eigenfunctions does not form a basis in Hilbert space* $H^1_{\mathrm{per}}([-\pi,\pi])$ *if* $\lim\limits_{n\to\infty} \cos(\widehat{f_n, f_{n+1}}) = 1$*.*

Proof. The Banach Theorem defines a condition that the complete and minimal set of eigenfunctions $\{f_n(\theta)\}_{n\in\mathbb{Z}}$ forms a basis in Hilbert space $H^1_{\mathrm{per}}([-\pi,\pi])$. According to the Theorem 2 on page 31 in

[3]By Lemma 8, all eigenvalues are simple starting with some $n \geq N$.

[14], the complete and minimal set of eigenfunctions forms a basis if and only if $\sup_N \|P_N\| < \infty$, where P_N is the projector of the linear span $\{f_n\}_{-N\le n\le N}$ in the direction of the linear span $\{f_n\}_{|n|\ge N+1}$. Since the Hilbert space $H^1_{\rm per}([-\pi,\pi])$ is a direct sum of the two linear spans above, the norm of the parallel projector P_N has the geometrical representation $\|P_N\| = \frac{1}{\sin\alpha_N}$, where α_N is the angle between the two linear spans [1]. This implies that the set $\{f_n(\theta)\}_{n\in\mathbb{Z}}$ is a basis in the Hilbert space $H^1_{\rm per}([-\pi,\pi])$ if

$$\cos(\widehat{f_n, f_{n+1}}) = \frac{|(f_n, f_{n+1})|}{\|f_n\|\|f_{n+1}\|} < 1, \tag{3.14}$$

for sufficiently large $n \in \mathbb{Z}$ [12]. In the other words, the angles between the eigenfunctions should be uniformly bounded away from zero as $n \to \infty$. If the angle tends to zero as $n \to \infty$, the set of eigenvectors is not a basis in the Hilbert space. □

4 Numerical approximations

We approximate isolated eigenvalues of the spectral problem (1.3) for $0 < \epsilon < 2$ numerically. In agreement with numerical results in [3], we show that all eigenvalues in the set $\{\lambda_n\}_{n\in\mathbb{Z}}$ are simple and purely imaginary. Therefore, the set $\{\lambda_n\}_{n\in\mathbb{Z}}$ can be ordered in the ascending order, such that $\lambda_0 = 0$, $\lambda_n = -\lambda_{-n}$, $\forall n \in \mathbb{N}$, $\mathrm{Im}\lambda_n < \mathrm{Im}\lambda_{n+1}$ and $\lim_{n\to\infty} |\lambda_n| = \infty$. We also show that the angle between two subsequent eigenfunctions $f_n(\theta)$ and $f_{n+1}(\theta)$ in the set $\{f_n(\theta)\}_{n\in\mathbb{Z}}$ tends to zero as $n \to \infty$.

4.1 Shooting method

The numerical shooting method is based on the ODE formulation of the spectral problem (1.3). By Lemma 7 and Corollary 1, complex eigenvalues $\lambda \in \mathbb{C}$ are determined by roots of the analytic function $F_\epsilon(\lambda)$ in the λ-plane. The number of complex eigenvalues can be computed with the winding number theory. The number and location of purely imaginary eigenvalues can be found from real-valued roots of a scalar real-valued function.

Proposition 1 *Let the eigenfunction $f(\theta)$ of the spectral problem (1.3) for $0 < \epsilon < 2$ be normalized by the condition $f(0) = 1$. The eigenvalue λ is purely imaginary if and only if $f(\theta) = \overline{f}(-\theta)$ on $\theta \in [-\pi,\pi]$.*

Proof. If $\lambda \in i\mathbb{R}$ and $f(\theta)$ satisfies the second-order ODE (1.3) on $\theta \in [-\pi,\pi]$, then $\bar{f}(-\theta)$ satisfies the same ODE (1.3) on $\theta \in [-\pi,\pi]$. By Corollary 1, if $f \in H^1_{\rm per}([-\pi,\pi])$, $f(0) = 1$ and $0 < \epsilon < 2$, the solution $f(\theta)$ is uniquely defined. By uniqueness of solutions, $f(\theta) = \overline{f}(-\theta)$ on $\theta \in [-\pi,\pi]$.

If $f(\theta) = \overline{f}(-\theta)$ on $\theta \in [-\pi,\pi]$, then,

$$\int_{-\pi}^{\pi} \sin\theta |f'(\theta)|^2 d\theta = \int_0^{\pi} \sin\theta |f'(\theta)|^2 d\theta - \int_0^{\pi} \sin\theta |f'(-\theta)|^2 d\theta = 0,$$

such that $\mathrm{Re}\lambda = 0$ according to the equality (2.10) in Lemma 4. □

Corollary 3 *Let $f(\theta)$ be an eigenfunction of the spectral problem (1.3) for $\lambda \in i\mathbb{R}$, such that $f \in H^1_{\rm per}([-\pi,\pi])$ and $f(0) = 1$. Then, $f(\pi) = f(-\pi)$ is equivalent to $f(\pi) \in \mathbb{R}$. The eigenvalue $\lambda \in i\mathbb{R}$ is simple if and only if*

$$(f^*, f) = 2{\rm Re} \int_0^{\pi} f(\theta)\bar{f}(\pi - \theta) d\theta \neq 0. \tag{4.1}$$

Proof. The first assertion follows by the symmetry relation $f(\theta) = \bar{f}(-\theta)$ evaluated at $\theta = \pi$. The second asserion follows by Lemma 3 with the use of the symmetry $f^*(\theta) = f(\pi - \theta)$. $\square$

Numerical Method: By using Lemma 7, the function $f(\theta)$ with $f(0) = 1$ is represented uniquely by the Frobenius series

$$f(\theta) = f_1(\theta) = 1 + \sum_{n \in \mathbb{N}} c_n \theta^n, \tag{4.2}$$

where the coefficients $\{c_n\}_{n\in\mathbb{N}}$ are uniquely defined by the recursion relation

$$c_n = -\frac{1}{n(1+\epsilon n)} \left(\lambda c_{n-1} + \epsilon n \sum_{m \in \mathbb{N}'} \frac{(-1)^{\frac{n-m}{2}} m}{(n-m+1)!} c_m \right), \quad n \in \mathbb{N}, \tag{4.3}$$

where $c_0 = 1$ and $\mathbb{N}'$ is a set of integers in the interval $[1, n-2]$ such that $n - m$ is even. For instance,

$$c_1 = -\frac{\lambda}{1+\epsilon}, \quad c_2 = \frac{\lambda^2}{2(1+\epsilon)(1+2\epsilon)}, \quad c_3 = -\frac{\lambda(\lambda^2 + \epsilon(1+2\epsilon))}{3!(1+\epsilon)(1+2\epsilon)(1+3\epsilon)},$$

and so on. We truncate the power series expansion on $N = 100$ terms and approximate the initial value $[f(\theta_0), f'(\theta_0)]$ at $\theta_0 = 10^{-8}$. By using the fourth-order Runge–Kutta ODE solver with time step $h = 10^{-4}$, we obtain a numerical approximation of $f \equiv f_+(\theta)$ on $\theta \in [\theta_0, \pi - \theta_0]$ for λ and $f \equiv f_-(\theta)$ on the same interval for $-\lambda$. By Lemma 3(i), the numerical approximation of the function $F_\epsilon(\lambda)$ of Corollary 1 is

$$\hat{F}_\epsilon(\lambda) = f_+(\pi - \theta_0) - f_-(\pi - \theta_0). \tag{4.4}$$

If $\lambda \in i\mathbb{R}$, the function $\hat{F}_\epsilon(\lambda)$ is simplified by using Corollary 3 as $\hat{F}_\epsilon(\lambda) = 2i{\rm Im} f_+(\pi - \theta_0)$. Table 1 represents the numerical approximations of the first four non-zero eigenvalues $\lambda \in i\mathbb{R}$ for $\epsilon = 0.5, 1.0, 1.5$[4] with the error computed from the residual

$$R = \left| \frac{(f, Lf)}{(f,f)} - \lambda \right|.$$

We can see from Table 1 that the accuracy drops with larger values of ϵ and for larger eigenvalues, but the eigenvalues persist inside the interval $|\epsilon| < 2$.

Figure 1 shows the profiles of eigenfunctions $f(\theta)$ on $\theta \in [0, \pi]$ for the first two eigenvalues $\lambda = i\omega_{1,2} \in i\mathbb{R}_+$ for $\epsilon = 0.5$ (left) and $\epsilon = 1.5$ (right). We can see from Fig. 1 that the derivative of $f(\theta)$ becomes singular as $\theta \to \pi^-$ for $\epsilon \geq 1$. We can also see that the real part of the eigenfunction $f(\theta)$ has one zero on $\theta \in (0, \pi)$ for the first eigenvalue and two zeros for the second eigenvalue, while the imaginary part of the eigenfunction $f(\theta)$ has a fewer number of zeros by one. The numerical approximations of

[4] We note that the Frobenius series (4.2) is not affected by the logarithmic terms for $\epsilon = 0.5$ and $\epsilon = 1.0$, since 0 is the largest index of the indicial equation at $\theta = 0$.

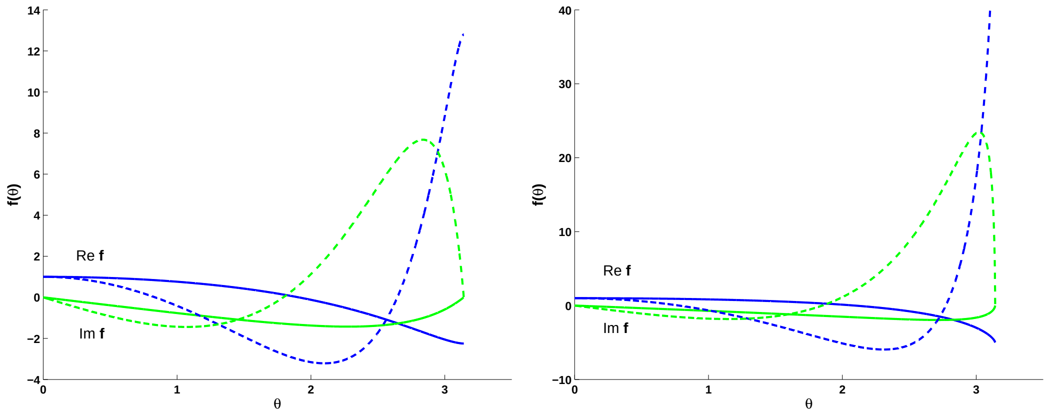

Figure 1: The real part (blue) and imaginary part (green) of the eigenfunction $f(\theta)$ on $\theta \in [0, \pi]$ for the first (solid) and second (dashed) eigenvalues $\lambda = i\omega_{1,2} \in i\mathbb{R}_+$ for $\epsilon = 0.5$ (left) and $\epsilon = 1.5$ (right).

the eigenvalue and eigenfunctions of the spectral problem (1.3) are structurally stable with respect to variations in θ_0, N and h.

Figure 2 shows the complex plane of $w = \hat{F}_\epsilon(\lambda)$ (left) and the argument of w (right) when λ traverses along the first quadrant of the complex plane $\lambda \in \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$ for $\epsilon = 0.5$. Here $\Lambda_1 = x + ir$ with $x \in [r, R]$, $\Lambda_2 = Re^{i\varphi}$ with $\varphi \in [\varphi_0, \frac{\pi}{2} - \varphi_0]$ and $\lambda_3 = r + iy$ with $y \in [r, R]$, where $r = 0.1$, $R = 10$, and $\varphi_0 = \arctan(r/R)$. It is obvious that the winding number of $\hat{F}_\epsilon(\lambda)$ across the closed contour is zero. Therefore, no zeros of $\hat{F}_\epsilon(\lambda)$ occurs in the first quadrant of the complex plane $\lambda \in \mathbb{C}$. The numerical result is structurally stable with respect to variations in r, R and ϵ.

ϵ	ω_1	R_1	ω_2	R_2	ω_3	R_3	ω_4	R_4
0.5	1.167342	0.000051	2.968852	0.000405	5.483680	0.001436	8.715534	0.003653
1.0	1.449323	0.000837	4.319645	0.007069	8.631474	0.024964	14.382886	0.061881
1.5	1.757278	0.002691	5.719671	0.018412	11.846709	0.054271	20.138824	0.113834

Table 1: Numerical approximations of the first four eigenvalues $\lambda = i\omega_n$ of the spectral problem (1.3) and the residuals $R = R_n$ for three values of ϵ.

4.2 Spectral method

The numerical spectral method is based on the reformulation of the second-order ODE (1.3) as the second-order difference equation and on the subsequent truncation of the difference eigenvalue problem. It is found in [16] that the truncation procedure lead to spurious complex eigenvalues which bifurcate off the imaginary axis.

Numerical method: Let $f(\theta)$ be an eigenfunction of the spectral problem (1.3) in $H^1_{\rm per}([-\pi, \pi])$. This eigenfunction is equivalently represented by the Fourier series

$$f(\theta) = \sum_{n \in \mathbb{Z}} f_n e^{-in\theta}, \qquad f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{in\theta} d\theta, \tag{4.5}$$

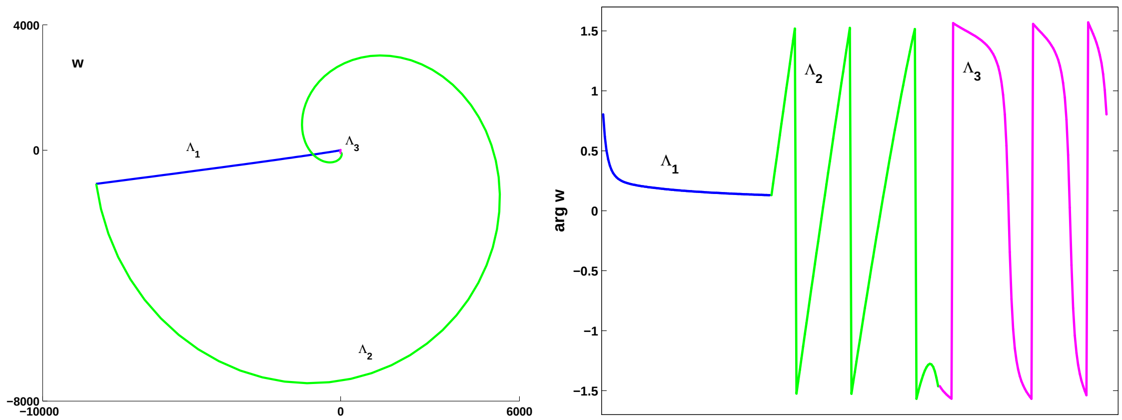

Figure 2: The image of the curve $w = \hat{F}_\epsilon(\lambda)$, when λ traverses along the contours Λ_1 (blue), Λ_2 (green) and Λ_3 (magenta) for $\epsilon = 0.5$: the image curve on the w-plane (left) and the argument of w (right).

where the infinite-dimensional vector $\mathbf{f} = (..., f_{-2}, f_{-1}, f_0, f_1, f_2, ...)$ is defined in $\mathbf{f} \in l_1^2(\mathbb{Z})$ equipped with the norm $\|\mathbf{f}\|^2_{l_1^2} = \sum_{n\in\mathbb{Z}}(1+n^2)|f_n|^2 < \infty$. The spectral problem (1.3) for $|\epsilon| < 2$ is equivalent to the difference eigenvalue problem

$$n f_n + \frac{\epsilon}{2} n \left[(n+1) f_{n+1} - (n-1) f_{n-1}\right] = -i\lambda f_n, \qquad n \in \mathbb{Z}. \tag{4.6}$$

The difference eigenvalue problem (4.6) splits into three parts

$$A\mathbf{f}_+ = -i\lambda \mathbf{f}_+, \qquad A\mathbf{f}_- = i\lambda \mathbf{f}_-, \qquad \lambda f_0 = 0, \tag{4.7}$$

where $\mathbf{f}_\pm = (f_{\pm 1}, f_{\pm 2}, ...)$ and A is an infinite-dimensional matrix

$$A = \begin{bmatrix} 1 & \epsilon & 0 & 0 & \cdots \\ -\epsilon & 2 & 3\epsilon & 0 & \cdots \\ 0 & -3\epsilon & 3 & 6\epsilon & \cdots \\ 0 & 0 & -6\epsilon & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{4.8}$$

Since $A = D - iS$, where D is a diagonal matrix and S is a self-adjoint tri-diagonal matrix, one can define the discrete counterpart of Lemma 4

$$\mathrm{Im}\lambda = \frac{(\mathbf{f}_+, D\mathbf{f}_+)}{(\mathbf{f}_+, \mathbf{f}_+)} = \frac{\sum_{n\in\mathbb{N}} n|f_n|^2}{\sum_{n\in\mathbb{N}} |f_n|^2}, \qquad \mathrm{Re}\lambda = \frac{(\mathbf{f}_+, S\mathbf{f}_+)}{(\mathbf{f}_+, \mathbf{f}_+)}.$$

where $\mathrm{Im}\lambda > 0$. The adjoint eigenfunction $f^*(\theta) = f(\pi - \theta)$ is recovered from the eigenvector $\mathbf{f}$ by $\mathbf{f}^* = J\mathbf{f}$, where

$$J = \begin{bmatrix} 0 & 0 & J_0 \\ 0 & 1 & 0 \\ J_0 & 0 & 0 \end{bmatrix}$$

and J_0 is a diagonal operator with entries $(-1, 1, -1, 1, ...)$.

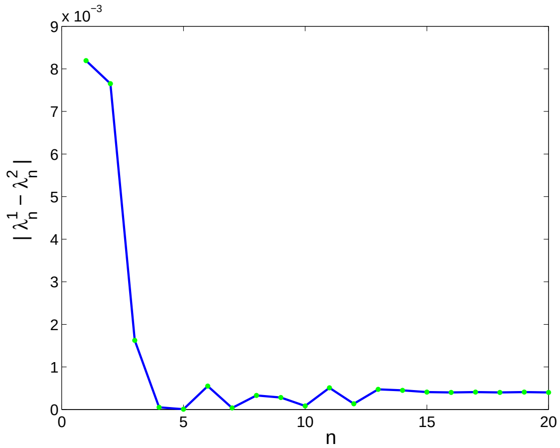

Figure 3: The distance between eigenvalues computed by the shooting and spectral methods for $\epsilon = 0.1$.

According to Theorem 1, rewritten from the set of eigenfunctions $\{f_n(\theta)\}_{n\in\mathbb{Z}}$ to the set of eigenvectors $\{\mathbf{f}_n\}_{n\in\mathbb{Z}}$, the inverse matrix operator A^{-1} is of the Hilbert-Schmidt type. Let A_N^{-1} denote the truncation of the matrix operator A^{-1} at the first N rows and columns. If a sequence of truncated operators A_N^{-1} converges uniformly to the limiting compact operator A^{-1} as $N \to \infty$, then the spectra of matrices A_N^{-1} also converge to the spectrum of A^{-1} as $N \to \infty$. However, the distance between eigenvalues of A_N^{-1} and A^{-1} may not be small for fixed N.

The smallest eigenvalues of the truncated matrix A_N^{-1} are found with the parallel Krylov subspace iteration algorithm [9]. Figure 3 shows the distance between eigenvalues of the shooting method and eigenvalues of the Krylov spectral method for $\epsilon = 0.1$. The difference between two eigenvalues is small of the order $\mathrm{O}(10^{-3})$ but the advantage of the parallel algorithm is that the calculating time of 20 largest eigenvalues of A_N^{-1} for $N = 10^6$ takes less than one minute on a network of 16 processors while finding the same set of eigenvalues by the shooting method with the time step $h = 10^{-5}$ takes about one hour.

Figure 4 shows symmetric pairs of eigenvalues of the matrix A_N for $\epsilon = 0.3$ at $N = 128$ (left) and $N = 1024$ (right). We confirm the numerical result of [16] that the truncation of the matrix operator A always produces splitting of large eigenvalues off the imaginary axis. Moreover, starting with some number n, the eigenvalues of A_N are real-valued. This feature is an artifact of the truncation, which contradicts to Lemmas 4 and 8 as well as to results of the shooting method. However, the larger is N, the more eigenvalues remain on the purely imaginary axis. Therefore, the corresponding eigenvectors can be used to compute the angle in Theorem 2.

Figure 5 (left) show the values of the cosine of the angle (3.14) for the first 20 purely imaginary eigenvalues for $\epsilon = 0.1$. As we can see from the figure, the angle between two eigenvectors tends to zero for larger eigenvalues up to the numerical accuracy. Figure 5 (right) and Table 2 show that the angle drops to zero faster with larger values of the parameter ϵ.

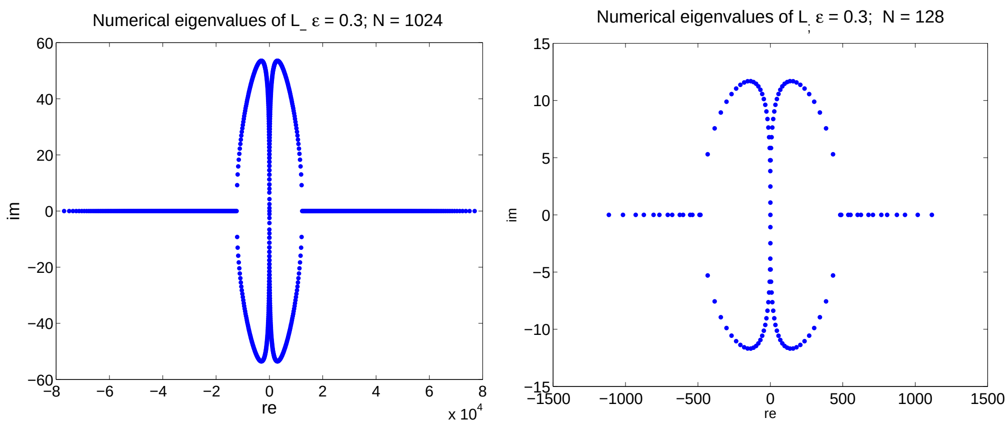

Figure 4: Spectrum of the truncated difference eigenvalue problem (4.6) for $\epsilon = 0.3$: $N = 128$ (left) and $N = 1024$ (right).

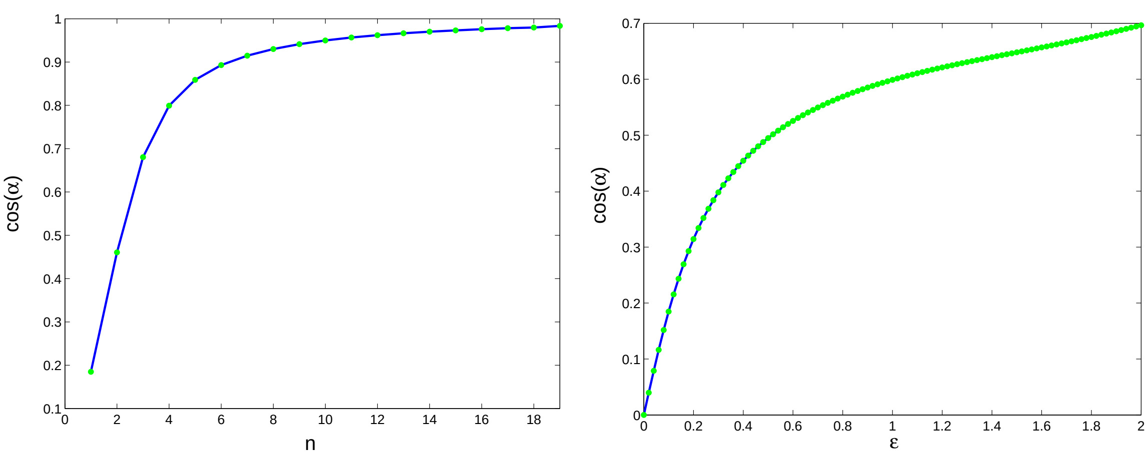

Figure 5: Left: the values of $\cos(\widehat{f_n, f_{n+1}})$ for the first 20 purely imaginary eigenvalues for $\epsilon = 0.1$. Right: the values of $\cos(\widehat{f_1, f_2})$ versus ϵ.

eigenvectors	$\epsilon = 0.1$	$\epsilon = 0.3$	$\epsilon = 0.5$
1-2	0.120166	0.325116	0.431987
2-3	0.461330	0.716192	0.780641
3-4	0.680709	0.838889	0.878055
4-5	0.799235	0.890440	0.914622
5-6	0.858944	0.921498	0.940306
6-7	0.892869	0.940395	0.955239
7-8	0.914745	0.953124	0.965235
8-9	0.930023	0.962120	0.972204
9-10	0.941262	0.968732	0.977265
10-11	0.949843	0.973741	0.981057
11-12	0.956580	0.977629	0.983988
12-13	0.961987	0.980702	0.986072
13-14	0.966407	0.983297	0.989617
14-15	0.970073	0.983459	0.990547
15-16	0.973153	0.995335	0.999101
16-17	0.975764	0.998749	0.999601

Table 2: Numerical values of $\cos(\widehat{f_n, f_{n+1}})$ for the first 16 purely imaginary eigenvalues for three values of ϵ.

The angle between two subsequent eigenvectors is closely related to the condition number

$$\|P_n\| = \frac{\|f_n\|\|f_n^*\|}{|(f_n, f_n^*)|} \equiv \mathrm{cond}(\lambda_n), \tag{4.9}$$

which measures the norm of the spectral projections [15]. By Lemma 3(iii), the condition number is infinite for multiple eigenvalues since $(f_n, f_n^*) = 0$. From the point of numerical accuracy, the larger is the condition number, the poorer is the structural stability of the numerically obtained eigenvalues to the truncation and round-off errors.

Figure 6 shows the condition number (4.9) computed for the first 40 purely imaginary eigenvalues for $\epsilon = 0.001$ and $\epsilon = 0.002$. We can see that the condition number grows for larger eigenvalues which indicate their structural instability. Indeed, starting with some number n, all eigenvalues are no longer purely imaginary. The condition numbers become extremely large with larger values of ϵ.

We finally illustrate that all true eigenvalues of the spectral problem (1.3) are purely imaginary and simple. To do so, we construct numerically the sign-definite imaginary type function and obtain the interlacing property of eigenvalues of the spectral problem (1.3) for two values $\epsilon = \epsilon_0$ and $\epsilon = \epsilon_1$, where $|\epsilon_1 - \epsilon_0|$ is small. We say that the eigenvalues exhibit the interlacing property if there exists an eigenvalue for $\epsilon = \epsilon_1$ between each pair of eigenvalues for $\epsilon = \epsilon_0$ and vice verse.

A meromorphic function $G(\lambda)$ is called a sign-definite imaginary type function if $\mathrm{Im}G(\lambda) \leq 0$ ($\mathrm{Im}G(\lambda) \geq 0$) on $\mathrm{Im}(\lambda) \leq 0$ ($\mathrm{Im}(\lambda) \geq 0$) [2]. We construct the meromorphic function $G(\omega)$ in the form $G(\lambda) = \frac{F_{\epsilon_0}(\lambda)}{F_{\epsilon_1}(\lambda)}$, where $F_\epsilon(\lambda)$ is an analytical function of Corollary 1. The numerical approximation of the meromorphic function $G(\lambda)$ is given by $\widehat{G}(\lambda) = \frac{\widehat{F}_{\epsilon_0}(\lambda)}{\widehat{F}_{\epsilon_1}(\lambda)}$. According to Theorems II.2.1 - II.3.1 on p. 437-439 in [2], the function $\widehat{G}(\lambda)$ is a meromorphic function of sign-definite imaginary type if and only

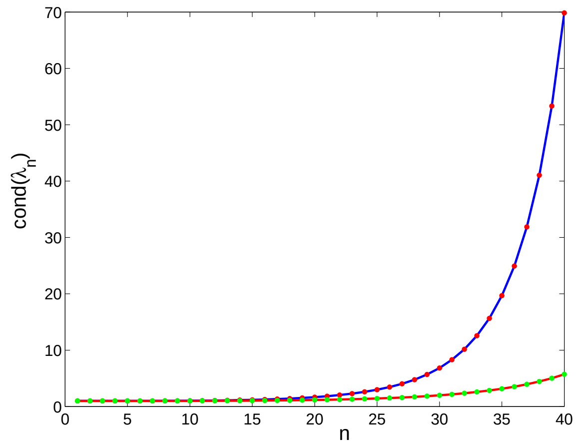

Figure 6: The condition number for the first 40 purely imaginary eigenvalues for $\epsilon = 0.001$ (red) and $\epsilon = 0.002$ (blue).

if it has the form $\widehat{G}(\lambda) = \frac{P(\lambda)}{Q(\lambda)}$ where $P(\lambda)$ and $Q(\lambda)$ are polynomials with real coefficients, with real and simple zeros, which are interlacing.

Table 3 shows this interlacing property of eigenvalues for $\epsilon_0 = 0.48$ and $\epsilon_1 = 0.5$. The remainder term $R_\epsilon = \frac{\|Lf-\lambda f\|}{\|\lambda f\|}$ measures the numerical error of computations. We have also computed numerically the values of $\widehat{G}(\lambda)$ on the grid $0.1 < \mathrm{Im}\lambda < 100$ and $0.1 < \mathrm{Re}\lambda < 100$ with step size 0.1 in both directions (not shown). Based on the numerical data, we have confirmed that the function $\widehat{G}(\lambda)$ does indeed belongs to the class of sign-definite imaginary type functions while the eigenvalues $\{\lambda_n\}_{n\in\mathbb{Z}}$ exhibit the interlacing property. This computation gives a numerical verification that all eigenvalues of the spectral problem (1.3) are simple and purely imaginary.

$\mathrm{Im}\lambda_{\epsilon_0}$	R_{ϵ_0}	$\mathrm{Im}\lambda_{\epsilon_1}$	R_{ϵ_1}
1.063112	$2.32438e-10$	1.068314	$2.40727e-10$
2.970880	$2.19667e-10$	3.024428	$2.25307e-10$
5.414789	$2.20243e-10$	5.542829	$2.26833e-10$
8.471510	$2.0904e-10$	8.693066	$2.15721e-10$
12.312548	$2.00793e-10$	12.665485	$2.06007e-10$
16.816692	$1.97653e-10$	17.327038	$2.0288e-10$
22.014084	$1.96165e-10$	22.711070	$2.01973e-10$
27.899896	$1.95265e-10$	28.812177	$2.01571e-10$
34.474785	$1.95008e-10$	35.631088	$2.01904e-10$
41.738699	$1.95577e-10$	43.167733	$2.03125e-10$
49.691673	$1.96711e-10$	51.422281	$2.04763e-10$
58.333258	$1.97959e-10$	60.391382	$2.06229e-10$
67.665387	$1.99038e-10$	70.140636	$2.0725e-10$
77.957871	$1.99894e-10$	79.828287	$2.0782e-10$
89.484519	$2.65662e-10$	91.544035	$2.08206e-10$

Table 3: The interlacing property of the first 15 purely imaginary eigenvalues for $\epsilon = 0.48$ and $\epsilon = 0.5$.

5 Conclusion

We have proved that the operator L associated with the heat equation (1.1) admits a closure in $L^2([-\pi,\pi])$ with a domain in $H^1_{\rm per}([-\pi,\pi])$ for $|\epsilon|<2$. The spectrum of L consists of isolated eigenvalues of finite multiplicities. Furthermore, we have proved with the assistance of numerical computations that the set of eigenfunctions of the spectral problem (1.3) for isolated eigenvalues is complete and minimal, but does not form a basis in $H^1_{\rm per}([-\pi,\pi])$. By using the analytic function theory and the truncated matrix eigenvalue problem, we have approximated the eigenvalues numerically and showed that all eigenvalues of the spectral problem (1.3) are purely imaginary.

We conclude therefore that the spectral problem (1.3) is not useful for direct analysis of well-posedness of the Cauchy problem for the heat equation (1.1). This is very unusual compared to the case of a standard heat equation

$$\left\{ \begin{array}{l} \dot{h} = -h_\theta - \epsilon h_{\theta\theta}, \quad t>0, \\ h(0) = h_0, \end{array} \right. \tag{5.1}$$

subject to the periodic boundary conditions on $\theta\in[-\pi,\pi]$. By using the method of separation of variables and the completeness of the Fourier series in $H^1_{\rm per}([-\pi,\pi])$, one can easily prove that there exists a unique solution of the heat equation (5.1) for any $h_0\in H^1_{\rm per}([-\pi,\pi])$ and any $\epsilon\in\mathbb{R}$ in the form

$$h(t) = \sum_{n\in\mathbb{Z}} c_n e^{\epsilon n^2 t} e^{in(x-t)}, \quad t\geq 0, \tag{5.2}$$

where the coefficients $\{c_n\}_{n\in\mathbb{Z}}$ are found in one and only one way from the initial data

$$\forall h_0\in H^1_{\rm per}([-\pi,\pi]): \quad h_0(\theta) = \sum_{n\in\mathbb{Z}} c_n e^{inx}. \tag{5.3}$$

If $\epsilon>0$, the Cauchy problem for the backward heat equation (5.1) is ill-posed. If h_0 is only in $H^1_{\rm per}([-\pi,\pi])$, coefficients $\{c_n\}_{n\in\mathbb{Z}}$ decays to zero algebraically fast as $n\to\infty$ and the solution (5.2) is singular for any $t>0$. Therefore, the solution $h(t)$ blows up in an infinitesimally small time.

If $\epsilon<0$, the Cauchy problem for the forward heat equation (5.1) is well-posed. In this case, the solution $h(t)$ becomes analytic in $\theta\in[-\pi,\pi]$ for any $t>0$ even if h_0 is only in $H^1_{\rm per}([-\pi,\pi])$. Therefore, there exists a constant $C>0$ such that $\|h(t)\|_{H^1_{\rm per}([-\pi,\pi])}\leq C\|h_0\|_{H^1_{\rm per}([-\pi,\pi])}$ for any $t\geq 0$.

Unlike this classical situation, the series of eigenfunctions of the spectral problem (1.3) are not applicable to construct a solution of the periodic heat equation (1.1), unless conditional convergence proposed in [4] can be adopted on a rigorous footing. Therefore, the PDE analysis of the Cauchy problem for the periodic heat equation (1.1) remains an open problem up to the date.

Note: When the project was essentially complete, we became aware of a preprint [7], where similar results were obtained. In particular, the author of [7] proves that the spectral problem (1.3) has no essential spectrum and, assisted with the numerical computations, illustrates that the eigenfunctions for all isolated eigenvalues do not form a basis. The analysis of [7] is based on the difference eigenvalue problem (4.6), which makes it different from our analysis.

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A Spectrum of the linear operator L in weighted spaces

The operator L can be rewritten in the Sturm-Liouville symmetric form

$$L = -\epsilon \left|\cot\left(\frac{\theta}{2}\right)\right|^{1/\epsilon} L_0, \quad L_0 = \frac{d}{d\theta}\left(\left|\tan\left(\frac{\theta}{2}\right)\right|^{1/\epsilon} \sin\theta \frac{d}{d\theta}\right). \tag{A.1}$$

Let $r(\theta) = |\tan(\theta/2)|^{1/\epsilon}$ be the weight of the Sturm–Liouville spectral problem

$$-\epsilon L_0 f(\theta) = \lambda r(\theta) f(\theta), \tag{A.2}$$

acting on smooth functions $f(\theta)$ on $\theta \in [-\pi, \pi]$ in the weighted space $f \in L^2_r([-\pi, \pi])$.

Proposition 2 *The operator L_0 admits a self-adjoint extension in $L^2_r([-\pi,\pi])$ for $0 < \epsilon < 1$, such that the spectrum of L_0 is purely discrete, consists of a set of simple real eigenvalues $\{\lambda_n\}_{n\in\mathbb{Z}}$ with $\lambda_0 = 0$, $\lambda_n = -\lambda_{-n}$, $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} \lambda_n = \infty$, and the eigenfunction $f_n(\theta)$ for $\lambda_n > 0$ is identically zero on $\theta \in [-\pi, 0]$.*

Proof. By Lemma 7, the eigenfunction $f(\theta)$ of the spectral problem (A.2) for $0 < \epsilon < 1$ is in $L^2_r([-\pi,\pi])$ if and only if it is spanned by the fundamental solutions $f_1(\theta)$, $f_2^+(\theta)$ and $f_2^-(\theta)$, such that $f(\theta)$ is bounded at $\theta = 0$ and $\lim_{\theta\to\pm\pi} f(\theta) = 0$. Let $f(\theta)$ on $\pm\theta \in [0, \pi]$ be represented by

$$f(\theta) = \left[\pm\cot\left(\frac{\theta}{2}\right)\right]^{1/2\epsilon} g_\pm(x), \qquad \cos\theta = x, \tag{A.3}$$

where $x \in [-1, 1]$. Then, the spectral problem (1.3) is rewritten in the form

$$-\epsilon \frac{d}{dx}\left[(1-x^2)\frac{dg_\pm}{dx}\right] + \frac{g_\pm(x)}{4\epsilon(1-x^2)} = \pm\frac{\lambda g_\pm(x)}{\sqrt{1-x^2}}. \tag{A.4}$$

If $f(\theta)$ belongs to $L^2_r([-\pi,\pi])$ for $0 < \epsilon < 1$, then $g_\pm(x)$ belong to $L^2_\rho([-1,1])$ with the weight function $\rho(x) = \frac{1}{\sqrt{1-x^2}}$. Since the symmetric spectral problem (A.4) is self-adjoint in space $L^2_\rho([-1,1])$, its eigenvalues λ are all real-valued. By Lemma 3 and the Sturm–Liouville theory [5], the eigenvalues $\{\lambda_n\}_{n\in\mathbb{Z}}$ are all simple and symmetric with $\lambda_0 = 0$ and $\lambda_n = -\lambda_{-n}$, $\forall n \in \mathbb{N}$, while the sequence $\{\lambda_n\}_{n\in\mathbb{Z}}$ is unbounded with $\lim_{n\to\infty} \lambda_n = \infty$. By the standard Green's identity,

$$\lambda \|g_+\|^2_{L^2_\rho} = \epsilon \int_{-1}^{1} (1-x^2)|g_+'(x)|^2 dx + \int_{-1}^{1} \frac{|g_+(x)|^2 dx}{4\epsilon(1-x^2)} > 0, \tag{A.5}$$

the eigenvalues λ_n of $g_+(x)$ are proved to be positive. If $\lambda_n > 0$ is an eigenvalue for $g_+(x)$, then $-\lambda_n < 0$ can not be an eigenvalue for $g_-(x)$, such that $g_-(x) = 0$ on $x \in [-1, 1]$. Therefore, $f_n(\theta) = 0$ on $\theta \in [-\pi, 0]$ if $\lambda_n > 0$. By Lemma 6, the spectrum of L is purely discrete for $|\epsilon| < 2$, such that the spectrum of L_0 in $L^2_r([-\pi,\pi])$ is also purely discrete. $\square$

Remark 5 The operator L_0 admits the same self-adjoint extension for any $\epsilon > 1$. However, this extension is not unique. Indeed, if $\epsilon > 1$, the eigenfunction $f(\theta)$ of the spectral problem (A.2) may exist in $L^2_r([-\pi,\pi])$ even if it is spanned by both fundamental solutions of Lemma 7 in the form

$$f(\theta) = Af_1(\theta) + Bf_2(\theta) = A^+ f_1^+(\theta) + B^+ f_2^+(\theta) = A^- f_1^-(\theta) + B^- f_2^-(\theta).$$

It follows from the Sturm–Liouville problem (A.2) that

$$\lambda \int_{-\pi}^{\pi} r(\theta)|f(\theta)|^2 d\theta = -\epsilon \left|\tan\left(\frac{\theta}{2}\right)\right|^{1/\epsilon} \sin(\theta)\bar{f}(\theta)f'(\theta)|_{\theta=-\pi}^{\theta=\pi} + \epsilon \int_{-\pi}^{\pi} \left|\tan\left(\frac{\theta}{2}\right)\right|^{1/\epsilon} \sin(\theta)|f'(\theta)|^2 d\theta.$$

The second integral is finite for $\epsilon > 1$ only if $B = 0$. In this case, the first term is computed explicitly for $\epsilon > 1$ as follows

$$2^{1/\epsilon}\left(\bar{A}^+ B^+ + \bar{A}^- B^-\right). \tag{A.6}$$

This term is zero if $A^+ = A^- = 0$ for the self-adjoint extension described in Proposition 2. However, this choice is not unique, e.g. the alternative pairing $B^+ = B^- = 0$ can also be applied.

B Resonant poles of the Schrödinger operators

Eigenvalues of the operator L coincide for $|\epsilon| > \frac{1}{2}$ with resonant poles of the Schrödinger operators. To show this, we use the transformation (3.5) on the intervals $\pm\theta \in [0,\pi]$ and rewrite the spectral problem (1.3) as the uncoupled spectral problems (3.6) for $f(\theta) \equiv f_\pm(t)$ on $t \in \mathbb{R}$. Let us apply the normalization condition $f(\pi) = f(-\pi) = 1$ for the eigenfunction $f(\theta)$ of the spectral problem (1.3) with $\lambda \notin \mathbb{R}$. Indeed, if either $f(\pi) = 0$ or $f(-\pi) = 0$, then the identity (A.5) implies that $\lambda \in \mathbb{R}$, but a real-valued λ can not be an eigenvalue of the spectral problem (1.3) by Lemma 4. Therefore, eigenfunctions $f_\pm(t)$ of the uncoupled problems (3.6) satisfy the boundary conditions

$$\lim_{t\to-\infty} f_\pm(t) = 1, \qquad \lim_{t\to\infty} f_\pm(t) = a^\pm, \tag{B.1}$$

where $a^\pm$ are uniquely defined. The function $f(\theta)$ on $\theta \in [-\pi,\pi]$ constructed from $f_\pm(t)$ on $t \in \mathbb{R}$ is continuous at $\theta = 0$ if $a^+ = a^-$.

Let $\epsilon > \frac{1}{2}$ and define $f_\pm(t) = e^{\frac{t}{2\epsilon}} g_\pm(t)$. The eigenfunctions $g_\pm(t)$ satisfy the linear Schrödinger equations

$$\left(\frac{1}{4\epsilon} - \epsilon\partial_t^2\right) g_\pm = \pm\lambda \operatorname{sech} t\, g_\pm, \tag{B.2}$$

but the boundary conditions (B.1) show that $g_\pm \notin L^2(\mathbb{R})$ and λ is not an eigenvalue. In fact, the eigenfunctions $g_\pm(t)$ belong to the exponentially weighted L^2-space, such that λ is a resonance pole of the Schrödinger operators. Let us decompose the eigenfunctions by $g_\pm(t) = e^{-\frac{t}{2\epsilon}} + h_\pm(t)$. By the theory of exponential asymptotics of solutions of the Schrödinger problems (B.2), it follows that $h_\pm \in L^2(\mathbb{R})$ if $\epsilon > \frac{1}{2}$. Let $h_0(t) = e^{-\frac{t}{2\epsilon}} \operatorname{sech} t \in L^2(\mathbb{R})$ and define the linear inhomogeneous problems for $h_\pm(t)$:

$$S_\epsilon^\pm(\lambda) h_\pm = \pm\lambda h_0(t), \qquad S_\epsilon^\pm(\lambda) = \frac{1}{4\epsilon} - \epsilon\partial_t^2 \mp \lambda \operatorname{sech} t. \tag{B.3}$$

The operator $S_\epsilon^\pm(\lambda)$ maps $H^2(\mathbb{R})$ to $L^2(\mathbb{R})$ and, if $\lambda \notin \mathbb{R}$, the kernel of $S_\epsilon^\pm$ is empty. The boundary condition $a^+ = a^-$ is then equivalent to the zeros of the function

$$H_\epsilon(\lambda) = \lambda \lim_{t\to\infty} e^{\frac{t}{2\epsilon}} \left[(S_\epsilon^+)^{-1}(\lambda) + (S_\epsilon^-)^{-1}(-\lambda)\right] h_0(t). \tag{B.4}$$

The function $H_\epsilon(\lambda)$ coincides (up to a multiplicative constant) with the analytic function $F_\epsilon(\lambda)$ introduced in Corollary 1 for $|\epsilon| < 2$. Therefore, $H_\epsilon(\lambda)$ represents a continuation of $F_\epsilon(\lambda)$ from the domain $|\epsilon| < 2$ to the domain $|\epsilon| > \frac{1}{2}$. The function $H_\epsilon(\lambda)$ is analytic in $\lambda \in \mathbb{C}$ and its roots give isolated eigenvalues of the spectral problem (1.3) with the account of their multiplicity.

The function $H_\epsilon(\lambda)$ can be simplified for $\lambda \in i\mathbb{R}$. Let $\lambda = i\omega \in i\mathbb{R}$ and represent $h_\pm = F(t) \pm iG(t)$, where

$$L_\epsilon F = -\omega \operatorname{sech} t\, G, \qquad L_\epsilon G = \omega \operatorname{sech} t\, F + \omega h_0(t), \qquad L_\epsilon = \frac{1}{4\epsilon} - \epsilon \partial_t^2. \tag{B.5}$$

Therefore, we can define a real-valued function $\tilde{H}_\epsilon(\omega) = H_\epsilon(i\omega)$ on $\omega \in \mathbb{R}$ given by

$$\tilde{H}_\epsilon(\omega) = \omega \lim_{t\to\infty} e^{\frac{t}{2\epsilon}} \left(L_\epsilon + \omega^2 \operatorname{sech} t\, L_\epsilon^{-1} \operatorname{sech} t \right)^{-1} h_0(t). \tag{B.6}$$

By multiplying the linear inhomogeneous equation

$$\left(L_\epsilon + \omega^2 \operatorname{sech} t\, L_\epsilon^{-1} \operatorname{sech} t \right) G = \omega h_0(t),$$

by $e^{\frac{t}{2\epsilon}}$ and integrating on $t \in \mathbb{R}$ by parts, the function $\tilde{H}_\epsilon(\omega)$ can be represented in the integral form

$$\tilde{H}_\epsilon(\omega) = \omega \int_{-\infty}^{\infty} \operatorname{sech} t dt - \omega^2 \int_{-\infty}^{\infty} e^{\frac{t}{2\epsilon}} \operatorname{sech} t\, L_\epsilon^{-1} \operatorname{sech} t G(t) dt.$$

Let $\tilde{h}_0(t) = \operatorname{sech} t\, L_\epsilon^{-1} \operatorname{sech} t e^{\frac{t}{2\epsilon}}$. Then,

$$\tilde{H}_\epsilon(\omega) = \pi\omega - \omega^2 \int_{-\infty}^{\infty} \tilde{h}_0(t) G(t) dt.$$

This is yet another form of the real analytic function, zeros of which are equivalent to purely imaginary eigenvalues of the spectral problem (1.3) for $|\epsilon| > \frac{1}{2}$.

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